CHAPTER

9 Definite Integration

The Fundamental Theorem of Calculus Part 1:

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Note: If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

✦ $\int_a^b f(x)dx$ = algebraic area under the curve $f(x)$ from a to b

Properties of Definite Integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\int_{-a}^a f(x)dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)dx, & f(-x)=f(x) \\ 0, & f(-x)=-f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x))dx$

6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$

7. $\int_0^{2a} f(x)dx = \int_0^a (f(x)+f(2a-x))\,dx$

$$= \begin{cases} 2\int_0^a f(x)dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

8. If $f(x)$ is a periodic function with period T, then

$\int_0^{nT} f(x)dx = n\int_0^T f(x)\,dx, n \in Z,$

$\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in Z, a \in R$

$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, m, n \in Z,$

$\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx, n \in Z, a \in R$

$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx, n \in Z, a, b \in R$

9. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$\int_a^b \Psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)dx$

Leibnitz Theorem

If $F(x) = \int_{g(x)}^{h(x)} f(t)dt,$

then $\dfrac{dF(x)}{dx} = h'(x) f(h(x)) - g'(x) f(g(x))$

Walli's Formula

1. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \dfrac{(n-1)(n-3)....(1 \text{ or } 2)}{n(n-2).....(1 \text{ or } 2)} K$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

2. $\int_0^{\pi/2} \sin^n x . \cos^m x \, dx$

$$= \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2} K$$

$$\text{where } K = \begin{cases} \frac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$$

Definite Integral as Limit of a Sum

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} h[f(a) + f(a+h) + f(a+2h) + ... + f(a + \overline{n-1}h)]$$

$$\Rightarrow \quad \lim_{h \to \infty} h \sum_{r=0}^{n-1} f(a+rh) = \int_0^1 f(x)dx \text{ where } b - a = nh$$

If $a = 0$ and $b = 1$ then, $\lim_{n \to \infty} h \sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$

OR $\lim_{n \to \infty} \left(\frac{1}{n}\right) \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx.$

Estimation of Definite Integral

1. If $f(x)$ is continuous in $[a, b]$ and it's range in this interval is $[m, M]$, them $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

2. If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

3. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx.$

4. If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x)dx \ge 0.$

5. $f(x)$ and $g(x)$ are two continuous function on $[a, b]$ then

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \le \sqrt{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}$$

Some Standard Results

1. $\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2 = \int_0^{\pi/2} \log \cos x \, dx$

2. $\int_a^b \frac{|x|}{x} dx = |b| - |a|.$